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THE MATRIX ALGEBRA FOR IMPLICATIONS

TWO developments of the "calculus of propositions,"—or algebra of implications,—may be said to give the same system provided the same propositions appear in each. Postulates of one development may be theorems of the other, and the same system may result from any number of different sets of assumptions. The system of material implication results from any set of postulates for the Boolean algebra, when the variables are interpreted as propositions.[1] The meaning of implication in this system is such that "p implies q" is not equivalent to "q can be inferred from p." In previous papers,[2] the writer has given postulates for another system, "Strict Implication," in which "p implies q" means exactly "q can be inferred from p."

The algebra to be presented in this paper is a new system, more comprehensive than any previously developed. The system of material implication, the system of strict implication, and at least one other system,—a calculus of consistencies which has never been developed,—are all included in it. We shall call it the "Matrix Algebra for Implications."[3] A set of assumptions for this system follows.

Primitive Ideas

The primitive ideas are the same as a possible set for material implication, with one additional idea,—impossibility (the truth value of a proposition which implies its own denial).

Propositions. $p, q, r,$ *etc.*, symbolize propositions or propositional functions. (A propositional function is an expression, involving a

[1] Postulates for this system have been given by Huntington, Müller (after Schröder), Peano, in Whitehead's "Universal Algebra," and in "Principia Mathematica," Whitehead and Russell. For the most economical set, see H. M. Sheffer, *Trans. Am. Math. Soc.*, Vol. XIV., pages 481–488.

[2] "A New Algebra of Implications, etc.," this JOURNAL, Vol. X., page 428, and "The Calculus of Strict Implications," *Mind*, N. S., No. 90.

[3] This designation is not intended to imply that this algebra is the most universal possible. Still more comprehensive systems for implication may be discovered.

variable or variables, which becomes a proposition when a value of the variable or variables is assigned.)

Negation. $-p$ symbolizes "not $-p$" or "p is false."

Impossibility. $\sim p$ symbolizes "p is impossible" or "It is impossible that p be true." The intent of this idea will become clearer as the system is developed.

Product. pq symbolizes "p and q both" or "p is true and q is true."

Equivalence. $p=q$ means "p is equivalent to q." This is the defining relation.

Truth Values

Systems previously developed have only two truth values,—truth and falsity. The addition of the idea "impossibility" gives us five truth values, all of which are familiar logical ideas.

1. $\sim p$. p is impossible; it is impossible that p be true.
2. $-p$. p is false.
3. $-\sim p$. p is possible; it is possible that p be true. Strictly, this should be written $-(\sim p)$. The parenthesis is regularly omitted for typographical reasons.
4. p. p is true.
5. $\sim -p$. p is necessary; it is impossible that p be false. The sign $\sim$ affects the whole of $-p$: parenthesis is regularly omitted, as in 3.[4]

The reader need be at no pains to grasp $-\sim p$ and $\sim -p$ as simple ideas. It is sufficient to understand $-p$ and $\sim p$ and to remember that each prefix affects the letter as already modified by those nearer it.

Definitions

The dyadic relations of propositions, with which the algebra principally deals, can be defined as truth values of products.

1. *Consistency.* $(p \circ q) = -\sim (pq)$. $\sim (pq)$, "It is impossible that p and q both be true," means "p and q are inconsistent." Hence its negative, $-\sim (pq)$, represents "p is consistent with q."

2. *Implication (inference).* $(p \supset q) = \sim (p{-}q)$.[5] "$p$ implies q

[4] It might be remarked that symmetry demands a sixth truth value. $-\sim p$ symbolizes strictly, not "p is possible," but "p is possibly true." We should, then, have "p is possibly false." In fact we do have it. $-\sim -p$ (each prefix affecting the whole of what follows it) is "It is false that it is impossible that p be false," *i. e.*, "p is possibly false." We may equally well observe that when "impossible" is distinguished from merely "false,"—as it is in ordinary logical thinking,—there are an indefinite, perhaps infinite, number of truth values. $-(-p)$ is equivalent to p, as will be postulated, but $\sim\sim p$ is irreducible, as is $-\sim\sim -p$, $\sim -\sim p$, $\sim\sim -p$, *etc.*

[5] We make use of symbols which have figured in "Principia Mathematica" and elsewhere *with different meanings.* The excuse for this is the availability of the types in question.

(q can be inferred from p)'' is here defined to mean ''It is impossible that p be true and q false.''

3. *Dilemmatic Disjunction.* $(p \vee q) = \sim(-p-q)$.

4. *Material Implication.* $(p < q) = -(p-q)$. $(p < q)$ means: ''It is false that p is true and q false.'' This is the implication relation of all developments of material implication. $(p < q)$ does *not* mean ''q can be inferred from p.'' When we speak of implication hereafter, the relation defined in 2 is always intended. When $(p < q)$ is meant, we shall write ''material implication'' and ''materially implies.''

5. *Non-dilemmatic Disjunction.* $(p+q) = -(-p-q)$. The difference between $(p+q)$ and $(p \vee q)$ is exactly analogous to that between $(p < p)$ and $(p \supset q)$. The distinction can not easily be preserved in English: both $(p \vee q)$ and $(p+q)$ would be rendered, ''Either p or q.'' It is probably the confusion of these two ideas which originally led to the development of material implication.[6]

The first three relations, $(p \circ q)$, $(p \supset q)$, and $(p \vee q)$, involve the idea of impossibility and do not belong to material implication. (pq), $(p < q)$, and $(p+q)$ do belong to material implication and do not involve the idea of impossibility. An interesting analogy between these two sets will appear shortly.

For the sake of completeness we add two definitions.

6. $(p \equiv q) = [(p < q)(q < p)]$.

7. $(p = q) = [(p \supset q)(q \supset p)]$. $(p \equiv q)$ is ''material equivalence'' (always so referred to in this paper), which is equivalence of truth value, not of intension. $(p = q)$, the defining relation, is itself defined, although a primitive idea, because this definition enables us to deduce other definitions. Obviously the analogy above referred to extends to $(p = q)$ and $(p \equiv q)$.

Postulates

P1. $(pq) \supset (qp)$.
P2. $(qp) \supset p$.
P3. $p \supset (pp)$.
P4. $[p(qr)] \supset [q(pr)]$.
P5. $[p \supset (q \supset r)] \supset [q \supset (p \supset r)]$.
P6. $(p \supset q) \supset [(q \supset r) \supset (p \supset r)]$.
P7. $p = -(-p)$.
P8. $(p \supset q) = (\sim q \supset \sim p)$.
P9. $\sim p \supset -p$.

In this set of postulates, economy and logical neatness have been somewhat sacrificed to clearness. If $(p+q)$ and the idea of necessary truth, $\propto p$, were taken as primitives, in place of (pq) and $\sim p$,

[6] See ''Implication and the Algebra of Logic,'' *Mind*, N. S., No. 84.

P7 could be proved, P8 reduced to an implication, and P4 and P5 replaced by a single postulate. But the ideas of $(p+q)$ and $\propto p$ are much harder to keep in mind than (pq) and $\sim p$. In the set as given, P7 might be reduced to $p \supset -(-p)$, but this would require several long and difficult proofs which we are able to omit with P7 as written.

P8 is equivalent to the pair $(p \supset q) \supset (- \sim p \supset - \sim q)$ and $(-p \supset -q) \supset (\sim p \supset \sim q)$. These propositions are more "self-evident" than the postulate, but express exactly the same relation.

Operations

Substitution.—Any proposition or propositional function may be substituted for *p, q, r, etc.,* in any primitive proposition or any theorem. Also, expressions which are equivalent,—$(p = q)$, not $(p \equiv q)$,—may be substituted for one another.

Inference.—If p is asserted and $(p \supset q)$ is asserted, q may be asserted.

Production.—If p and q are separately asserted, (pq) may be asserted. (This operation is not indispensable, but occasionally renders proof much less cumbersome.)

The first theorem will be proved in full. Later proofs will be abbreviated or indicated only.

Theorem 1. $(pq) \supset p.$

P6. $\{(pq)/p;\ (qp)/q;\ p/r\}$: (P1) $\supset \{$ (P2) $\supset [(pq) \supset p]$.
This proof may be read: "Postulate 6; when (pq) is substituted for p; (qp) for q; and p for r; states that: Postulate 1 implies that Postulate 2 implies $(pq) \supset p$." Let us make the substitutions indicated in Postulate 6. We then have

$$[(pq) \supset (qp)] \supset \{[(qp) \supset p] \supset [(pq) \supset p].$$

The expression in the first pair of brackets is P1. Since P1 is asserted, its implication, which follows it, may be asserted. The expression in the second pair of brackets is P2. Since P2 is asserted, its consequence, Theorem 1, may be asserted.

Th. 2. $(p \supset q) \supset (\sim q \supset \sim p).$

By Def. 7,

$(P8) = \{[(p \supset q) \supset (\sim q \supset \sim p)] [(\sim q \supset \sim p) \supset (p \supset q)]\}$ (a)

Th. 1; (a) $\supset$ Q. E. D.

We introduce here abbreviations of proof as follows: (a), or (b), etc., is placed after a lemma which has been established, and thereafter in the same proof we write (a), or (b), etc., instead of that lemma. Also, we shall frequently write "Q. E. D." in the last line of proof in place of the theorem to be proved. In the second line of

this proof, the substitutions which it is necessary to make in Th. 1 in order to get (a) $\supset [(p \supset q) \supset (\sim q \supset \sim p)]$ are not stated because they are obvious. If (a) is pq, p is clearly Th. 2.

Theorem 2 is one of the implications contained in P8. By Def. 7, a pair of implications may be substituted for any equivalence. By P2 and Th. 1, either of these implications may be taken separately.

Th. 3. $(-p \supset q) \supset (-q \supset p)$.
P1 $\{-q/p; -p/q\}$: $(-q-p) \supset (-p-q)$ (a)
Th. 2 $\{(-q-p)/p; (-p-q)/q\}$: (a) $\supset [\sim(-p-q) \supset \sim(-q-p)$
by Def. 2, $(-p \supset q) \supset (-q \supset p)$.

Th. 4. $(p \supset q) \supset (-q \supset -p)$.
Th. 3 $\{-p/p\}$: $[-(-p) \supset q] \supset [(-q \supset -p)]$.
by P7, $(p \supset q) \supset (-q \supset -p)$.

Th. 5. $(p \supset -q) \supset (q \supset -p)$.

Similar proof.

Th. 6. $(-p \supset -q) \supset (q \supset p)$.

Similar proof.

Theorems 3, 4, 5, and 6 are the four forms of the principle of permutation. Use of any one of these theorems is indicated hereafter by "perm."

Th. 7. $(p \circ q) \supset (q \circ p)$.
Th. 2 $\{(pq)/p; (qp)/q\}$: (P1) $\supset [\sim(qp) \supset \sim(pq)]$ (a).
Perm. (a) $\supset [-\sim(pq) \supset -\sim(qp)]$
by Def. 1, $(p \circ q) \supset (q \circ p)$.

Th. 8. $(q \circ p) \supset -\sim p$.
Th. 2 $\{(qp)/p; p/q\}$: (P2) $\supset [\sim p \supset \sim(qp)]$ (a).
Perm. (a) $\supset [-\sim(qp) \supset -\sim p]$
by Def. 1, $(q \circ p) \supset -\sim p$.

Th. 9. $-\sim p \supset (p \circ p)$.

Similar proof, using P3.

Th. 10. $[p \circ (qr)] \supset [q \circ (pr)]$.

Similar proof, using P4.

Th. 11. $[p \circ (q \circ r)] \supset [q \circ (p \circ r)]$.
P5 $\{q/p; p/q; -r/r\}$: $[q \supset (p \supset -r)] \supset [p \supset (q \supset -r)]$
by Def. 2, $\{q \supset \sim[p-(-r)]\} \supset \{p \supset \sim[q-(-r)]\}$
by P7, $[q \supset \sim(pr)] \supset [p \supset \sim(qr)]$
by Def. 2, $\sim[q-\sim(pr)] \supset \sim[p-\sim(qr)]$ (a).
Perm. (a) $\supset \{-\sim[p-\sim(qr)] \supset -\sim[q-\sim(pr)]$
by Def. 1, $[p \circ (q \circ r)] \supset [q \circ (p \circ r)]$.

We can now exhibit the analogy between products and consistencies, facts and possibilities, which runs all through the system.

P1. $(pq) \supset (qp)$.	Th. 7. $(p \circ q) \supset (q \circ p)$.
P2. $(qp) \supset p$.	Th. 8. $(q \circ p) \supset - \sim p$.
P3. $p \supset (pp)$.	Th. 9. $- \sim p \supset (p \circ p)$.
P4. $[p(qr)] \supset [q(pr)]$.	Th. 11. $[p \circ (q \circ r)] \supset [q \circ (p \circ r)]$.[7]

The same analogy holds between non-dilemmatic and dilemmatic disjunctions, as follows:

$(p + q) \supset (q + p)$	$(p \vee q) \supset (q \vee p)$
$p \supset (p + q)$	$\sim - p \supset (p \vee q)$
$(p + p) \supset p$	$(p \vee p) \supset \sim - p$
$[p + (q + r)] \supset [q + (p + r)]$	$[p \vee (q \vee r)] \supset [q \vee (q \vee r)]$

We prove the two theorems given in the first line to illustrate the method by which theorems involving disjunctions are deduced.

Th. 12.	$(p + q) \supset (q + p)$.	
P1 $\{-q/p;\ -p/q\}$:	$(-q - p) \supset (-p - q)$	(a).
Perm.	(a) $\supset [- (-p - q) \supset - (-q - p)$	
by Def. 5,	$(p + q) \supset (q + p)$.	

Th. 13.	$(p \vee q) \supset (q \vee p)$.	
P1 $\{-q/p;\ -p/q\}$:	$(-q - p) \supset (-p - q)$	(a).
P9.	(a) $\supset [\sim (-p - q) \supset \sim (-q - p)$	
by Def. 3,	$(p \vee q) \supset (q \vee p)$.	

Again, the same analogy holds between material implications and implications of the type of inference:

$(-p < q) \supset (-q < p)$	Th. 2. $(-p \supset q) \supset (-q \supset p)$
$-p \supset (p < q)$	$\sim p \supset (p \supset q)$
$(p < -p) \supset -p$	$(p \supset -p) \supset \sim p$
$p \supset (q < p)$	$\sim - p \supset (q \supset p)$
$(-p < p) \supset p$	$(-p \supset p) \supset \sim - p$

Note that the *main* implication in theorems in both columns is of the type of inference.

We prove the second theorem in each column.

Th. 14.	$-p \supset (p < q)$.	
Th. 1 $\{-q/q\}$:	$(p - q) \supset p$	(a).
Perm.	(a) $\supset [-p \supset - (p - q)$	
by Def. 4,	$-p \supset (p < q)$.	

[7] If P4 could be written $(pqr) \supset (qpr)$, an exact analogue $(p \circ q \circ r) \supset (q \circ p \circ r)$, could be derived from it. As it is, P4 proves Th. 10, and Th. 11 is derived from P5.

Th. 15.	$\sim p \supset (p \supset q)$.	
Th. 1 $\{-q/q\}$:	$(p - q) \supset p$	(a).
Th. 2.	(a) $\supset [\sim p \supset \sim (p - q)$	
by Def. 2,	$\sim p \supset (p \supset q)$.	

The Calculus of Consistencies

This persistent analogy between the relations which figure in material implication, pq, $p + q$, and $p < q$, and the set involving the idea of impossibility, $p \circ q$, $p \vee q$, and $p \supset q$, suggests that there may be a calculus of possibilities and impossibilities, similar to the system of material implication. We give below assumptions sufficient for such a system, which might be called the calculus of consistencies.

Primitive Ideas.—Propositions, negation, equivalence, and consistency,—all symbolized as before.

Definitions:

$$(a) \quad (p \vee q) = -(-p \circ -q)$$
$$(b) \quad (p \supset q) = (-p \vee q)$$

Postulates.—These postulates will be lettered, *a, b,* . . . to avoid confusion with the postulates of the matrix algebra. For economy of assumption, they will be expressed in terms of the relation $\vee$. But since this relation is easily confused with $+$, we give *a'*, *b'*, . . ., equivalent respectively to *a, b*. . . .

(*a*)	$(p \vee q) \supset (q \vee p)$.	(*a'*)	$(p \circ q) \supset (q \circ p)$.
(*b*)	$(p \vee p) \supset (p \vee q)$.	(*b'*)	$(p \circ q) \supset (p \circ p)$.
(*c*)	$(p \vee p) \supset p$.	(*c'*)	$p \supset (p \circ p)$.
(*d*)	$[p \vee (q \vee r)] \supset [q \vee (p \vee r)]$.	(*d'*)	$[p \circ (q \circ r)] \supset [q \circ (p \circ r)]$.
(*e*)	$(q \supset r) \supset [(p \vee q) \supset (p \vee r)]$.	(*e'*)	$(q \supset r) \supset [(p \circ q) \supset (p \circ r)]$.

The force of *b* and *c* may not at once be clear. *B*, or *b'*, expresses the assumption that if *p* is consistent with some (any) other proposition, *q*, it is self-consistent or non-contradictory. This is equivalent to $-(p \circ p) \supset -(p \circ q)$, "A proposition not self-consistent is inconsistent with any other," and to $(-p \supset p) \supset (q \supset p)$, "A proposition implied by its own negation is implied by any (every) proposition." *C*, or *c'*, states that every true proposition is self-consistent.

This calculus has never been developed. It is not identical with Mrs. Ladd-Franklin's algebra[8] based on the notions of consistency and inconsistency, but has interesting relations to that system. All the propositions of the calculus of consistencies can be proved from the assumptions of the matrix algebra for implications.

[8] In "Studies in Logic by Members of Johns Hopkins University," ed., Pierce.

Material Implication

The propositions of the system of material implication also can be proved from the assumptions of the matrix algebra. We proceed to deduce the primitive propositions for material implication given in "Principia Mathematica."[9]

The primitive ideas; p, q, r, etc.; $-p$; and $(p=q)$; belong to material implication as to the matrix algebra. The other primitive is "either p is true or q is true, where the alternatives are to be not mutually exclusive" (p. 97). The use of this relation shows it to be non-dilemmatic disjunction, which we symbolize, $p+q$.

Of the definitions, one, $(p \equiv q) = [(p < q)(q < p)]$, is already assumed in the matrix algebra. The other two may be deduced. (Primitive propositions and theorems used in the proofs are, of course, those of the matrix algebra.)

Th. 16. $(p < q) = (-p + q)$ [10]
Def. 4. $(p < q) = -(p - q)$.
Def. 5. $\{-p/p\}$: $(-p + q) = -[-(-p) - q)$
by P7, $(-p + q) = -(p - q)$
by substitution, $(p < q) = (-p + q)$.

For the deduction of the next definition, three lemmas must first be proved.

Th. 17. $(p = q) \supset (p \supset q)$.
Def. 7 $\{(p=q)/p;\ [(p \supset q)(q \supset p)]/q\}$:
$$(\text{Def. 7}) = [\{(p=q) \supset [(p \supset q)(q \supset p)]\} \{[(p \supset q)(q \supset p)] \supset (p=q)\}] \quad \text{(a)}.$$

(The last three lines should be read as one.)

Th. 1. (a) $\supset \{(p=q) \supset [(p \supset q)(q \supset p)]$ (b).
Th. 1. $[(p \supset q)(q \supset p)] \supset (p \supset q)$ (c).
P6. (b) $\supset$ [(c) $\supset$ Q. E. D.

Th. 18. $(p=q) \supset (q \supset p)$.
Similar proof.

Th. 19. $[(p \supset q)(q \supset p)] \supset (p=q)$.
P2. [(a) in proof of Th. 17] $\supset$ Q. E. D.

Th. 20. $(pq) = -(-p + -q)$ [*Prin. Math.*, *3.01].
Def. 5 $\{-p/p;\ -q/q\}$: $(-p + -q) = -[-(-p) - (-q)]$
by P7, $(-p + -q) = -(pq)$ (a).
Th. 17. (a) $\supset [(-p + -q) \supset -(pq)]$ (b).
Th. 18. (a) $\supset [-(pq) \supset (-p + -q)]$ (c).

[9] Pp. 95–101, 114, and 120. We neglect those refinements of the development given in "Principia Mathematica" which are due to the theory of types.

[10] "Prin. Math.," *1·01.

Perm. $(b) \supset [(pq) \supset -(-p + -q)]$ (d).
Perm. $(c) \supset [-(-p + -q) \supset (pq)]$ (e).
by production of (d) and (e),
$\{[(pq) \supset -(-p + -q)][-(-p + -q) \supset (pq)]\}$ (f).
Th. 19. $(f) \supset$ Q. E. D.

We proceed to the proof of the postulates as given in "Principia Mathematica." The first step is the derivation of theorems similar to these postulates, but not identical with them.

Th. 21. $(p + p) \supset p.$
P3 $\{-p/p\}$: $-p \supset (-p-p)$ (a).
Perm. $(a) \supset [-(-p-p) \supset p$
by Def. 5, $(p + p) \supset p.$

Th. 22. $q \supset (p + q).$
P2 $\{-p/q;\ -q/p\}$: $(-p-q) \supset -q$ (a).
Perm. $(a) \supset [q \supset -(-p-q)$
by Def. 5, $q \supset (p + q).$

Th. 23. $(p + q) \supset (q + p).$

Similar proof, using P1.

Th. 24. $[p + (q + r)] \supset [q + (p + r)].$
P4 $\{-q/p;\ -p/q;\ -r/r\}$: $[-q(-p-r)] \supset [-p(-q-r)]$ (a).
Perm. $(a) \supset \{-[-p(-q-r)] \supset -[-q(-p-r)]$
by P7, $-\{-p-[-(-q-r)]\} \supset -\{-q-[-(-p-r)]\}$
by Def. 5, $[p + -(-q-r)] \supset [q + -(-p-r)]$
by Def. 5, $[p + (q + r)] \supset [q + (p + r)].$

Th. 25. $(\sim p \supset \sim q) \supset (q \supset p).$
Th. 18. $(P8) \supset$ Q. E. D.

Th. 26. $(\sim p \supset \sim q) \supset (-p \supset -q).$
Perm. $(q \supset p) \supset (-p \supset -q)$ (a).
P6. $(a) \supset$ [(Th. 25) $\supset$ Q. E. D.

Th. 27. $[(p \supset q) \supset (r \supset s)] \supset [(p < q) \supset (r < s).$
Th. 26 $\{(p-q)/p;\ (r-s)/q\}$:
$[\sim(p-q) \supset \sim (r-s)] \supset [-(p-q) \supset -(r-s)]$
by Defs. 2 and 4, Q. E. D.

Th. 28. $(p \supset q) \supset (p < q).$
P9 $\{(p-q)/p\}$: $\sim (p-q) \supset -(p-q)$
by Defs. 2 and 4, Q. E. D.

Th. 29. $(q \supset r) \supset [(p < q) \supset (p < r)].$
P5. $(P6) \supset \{(q \supset r) \supset [(p \supset q) \supset (p \supset r)]\}$ (a).
P6. $(a) \supset$ [(Th. 25) $\supset$ Q. E. D.

Th. 30. $(q \supset r) \supset [(p+q) \supset (p+r)]$.
Th. 29 $\{-p/p\}$: $(q \supset r) \supset [(-p < q) \supset (-p < r)]$
by Def. 4, $(q \supset r) \supset [-(-p-q) \supset -(-p-r)]$
by Def. 5, Q. E. D.

Th. 31. $(q < r) \supset [(p+q) < (p+r)]$.
Th. 27 $\{q/p;\ r/q;\ (p+q)/r;\ (p+r)/s\}$: (Th. 30) $\supset$ Q.E.D.

We now have a set of theorems which differ from the postulates for material implication in only one respect,—each of these theorems contains one implication of the type $\supset$. But Th. 28 tells us that from any implication, $(p \supset q)$, the corresponding material implication, $(p < q)$, can be inferred. By use of this theorem we prove immediately:

Th. 32.	$(p+p) < p$[11]	(from Th. 21).
Th. 33.	$q < (p+q)$[12]	(from Th. 22).
Th. 34.	$(p+q) < (q+p)$[13]	(from Th. 23).
Th. 35.	$[p+(q+r)] < [q+(p+r)]$[14]	(from Th. 24).
Th. 36.	$(q < r) < [(p+q) < (p+r)]$[15]	(from Th. 31).

These are the postulates for material implication. However, the system of material implication, as previously developed, requires an operation which may be stated: "If p is asserted and $(p < q)$ is asserted, q may be asserted."[16] We do not assume this operation, because $(p < q)$ is not equivalent to "q can validly be inferred from p." But we can prove all the theorems of material implication from the postulates of the matrix algebra in much the same way as we have just proved the postulates. Also, by using the postulates and theorems of the matrix algebra,—*e. g.*, P6 and Th. 27,—as principles of inference, we can prove that the theorems of material implication *can be inferred from* the postulates of material implication. This has never before been shown. Previous developments of material implication have proceeded by means of numerous "mathematical operations" or have proved only that the theorems are *materially implied by* the postulates.

If, however, the opinion expressed in this paper, that the relation of material implication is not equivalent to valid inference, be correct, then the system of material implication has no value as an organon of proof, and its interest is chiefly mathematical and historical.

[11] "Prin. Math.," *1.2.
[12] *Ibid.*, *1.3.
[13] *Ibid.*, *1.4.
[14] *Ibid.*, *1.5.
[15] *Ibid.*, *1.6.
[16] See "Prin. Math.," *1.1.

Strict Implication

The following set of definitions and postulates is sufficient for the system of Strict Implication.[17]

S1. $(p \supset q) = (-p \vee q)$.
S2. $(pq) = -(-p + -q)$.
S3. $(p \vee q) \supset (q \vee p)$.
S4. $(p + q) \supset (q + p)$.
S5. $(p + p) \supset p$.
S6. $q \supset (p + q)$.
S7. $(p \vee q) \supset (p + q)$.
S8. $[p \vee (q \vee r)] \supset [q \vee (p \vee r)]$.
S9. $[p \vee (q + r)] \supset [q \vee (p + r)]$.
S10. $(q \supset r) \supset [(p \vee q) \supset (p \vee r)]$.
S11. $(q \supset r) \supset [(p + q) \supset (p + r)]$.

S2 is Th. 20; S3 is Th. 13; S4 is Th. 12; S5 is Th. 21; S6 is Th. 22; S11 is Th. 30. We proceed to proof of the remainder.

Th. 37. $(p \supset q) = (-p \vee q)$. [S1]
Def. 2, $(p \supset q) = \sim (p - q)$
by P7, $(p \supset q) = \sim [-(-p) - q]$.
Def. 3, $(-p \vee q) = \sim [-(-p) - q)$
by substitution, Q. E. D.

Th. 38. $(p \vee q) \supset (p + q)$. [S7]
P9 $\{(-p - q)/p\}$: $\sim(-p - q) \supset -(-p - q)$
by Defs. 3 and 5, Q. E. D.

Th. 39. $[p \vee (q \vee r)] \supset [q \vee (p \vee r)]$ [S8]
P5 $\{-p/p; -q/q\}$: $[-p \supset (-q \supset r)] \supset [-q \supset (-p \supset r)]$
by Th. 37, $\{-p \supset [-(-q) \vee r]\} \supset \{-q \supset [-(-p) \vee r]\}$
by Th. 37, $\{-(-p) \vee [-(-q) \vee r]\} \supset \{-(-q) \vee [-(-p) \vee r]\}$
by P7, Q. E. D.

Th. 40. $[p \vee (q + r)] \supset [q \vee (p + r)]$. [S9]
Th . 2 $\{-q/p; -p/q; -r/r\}$:
$(P4) \supset \{\sim[-p(-q-r)] \supset \sim[-q(-p-r)]$
by P7, $\sim\{-p-[-(-q-r)]\} \supset \sim\{-q-[-(-p-r)]\}$
by Def. 3, $\{p \vee [-(-q-r)]\} \supset \{q \vee [-(-p-r)]\}$
by Def. 5, Q. E. D.

Th. 41. $(q \supset r) \supset [(p \vee q) \supset (p \vee r)]$. [S10]
P5. $(P6) \supset \{(q \supset r) \supset [(p \supset q) \supset (p \supset r)]\}$ (a);
(a) $\{-p/p\}$: $(q \supset r) \supset [(-p \supset q) \supset (-p \supset r)]$
by Th. 37, $(q \supset r) \supset \{[-(-p) \vee q] \supset [-(-p) \vee r]$
by P7, Q. E. D.

[17] See "The Calculus of Strict Implication," *Mind*, N. S., No. 90, page 243.

Thus the entire system of strict implication is included in the matrix algebra.

Aside from its comprehensive character, the matrix algebra for implications is useful as an instrument for investigating the interrelations of necessity, truth, possibility, falsity, and impossibility, and of such derivative relations as have figured in this paper. If the assumptions of this system are true, the consequences are important not only for logic, but also for epistemology and metaphysics.

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ARE REALISM AND RELATIVITY INCOMPATIBLE?

"THE new realism," says Professor Lovejoy in a recent article in this JOURNAL,[1] "from the first had in it a strain of relativism." "Yet a thinker of the true realistic temperament," he goes on to say, "craves a world of objects which have each *some* intrinsic and solid character, which do not endlessly deliquesce into mere relations to other things, themselves equally characterless and elusive. . . . But it appears to be the fact that practically these two incongruous motives were conjoined in the production of neo-realism in its American form."

I dare say that many of us are ready to subscribe to Professor Lovejoy's contentions as here expressed. I must differ with him, however, if he holds that the realist's craving for solid *things* and the principle of relativity are inherently "two incongruous motives."

It is true that realism is prone to ignore a dependence among objects and that relativity is disposed to the opposite error of discounting an independence among them. But relativity, properly conceived, must include a specific investigation into the nature of its terms just as realism finds and has found it incumbent to recognize the effect of relations. Hence, whether for a so-called realist or for a relativist, the conjunction of things and relations appears inevitable. It is easy to perceive why a conjunction between them may stand obscured and denied; for relativity, when brought to its formulation, is as commonly lacking in its account of *things* as realism is lacking in a precise definition of its special stumbling-block,—the nature and scope of relations. Many, I have no doubt, would be surprised to find to what extent John Locke has already struggled, with no mean success, to solve this very issue. In Book III. of his Essay (to paraphrase him as nearly as possible in his

[1] Vol. XI., page 421.

own language) he writes: objects exist in nature with a measure and boundary that appear prefixed; yet gold in one relation is solid and in another liquid, in one relation yellow and in another red, green, or what not. Moreover, men do not appear to agree in their conception of the simplest and most common of objects. How, then, do objects acquire such measure and boundary as they possess? Having presented this matter at large in two publications,[2] I shall merely revert to what I think is pertinent in those publications to the issue in question. In them I have attempted a complete formulation of relativity.

Since the crucial issue for relativity centers in a definition of *things,* we may state in advance that those formulations of relativity that terminate in a mere flux-doctrine or in the more widely current one that objects "endlessly deliquesce into mere relations," are doctrines that have hopelessly lost sight of their most difficult problem. A relativist's point of departure is the dynamic world; but realism, with its apotheosis of a static world, rises to overwhelm him at the very point where a relativist's chain of reasoning is invariably at its weakest—its impotence in the face of what is static. Independence is a characteristic of things, writes the realist. Let us see to what extent this realistic doctrine can compel a relativist to rescind the inadequate doctrine that objects "endlessly deliquesce into mere relations."

Underlying this particular formulation of relativity is the abstract conception that each object is involved in all and all in each. This sounds beautiful, but what basis has the conception? The conception emphatically denies on its very face that a partiality or an indifference exists in the effective relations of one object with another. But the principle of elimination as involved in our inductive methods (to cite but one instance) declares as emphatically at the outset that *disconnections* exist among things, so overwhelming and obtrusive in their presence, as to make the search for *connections* among some of them the most arduous object of a scientist's special endeavor. In fact, if no real disconnections exist then neither do connections exist, for the two ideas involve and presuppose each other at every point, and neither of them can claim a reality more ultimate than the other. It is thus that we come to recognize that the dependence of a given object in a given (circumscribed) situation may be large, yet the dependence of other objects upon it or upon each other in that particular situation may be zero. The unaffected objects are accordingly more properly designated as inde-

[2] "Locke a Constructive Relativist," The Scientific Press, Brooklyn, 1912. "A New Conception of Relativity and Locke," University of Cincinnati Studies, 1914.

pendent (in that situation). But an independence properly maintained for an object in certain situations may in other situations convert itself into a mere dependence, as our scientific postulate of a general dependence of an object would naturally dispose us to expect. This is the first significant distinction,—obvious, but abundantly overlooked, that I would insist upon; namely, that to talk of an object's general dependence, as a dependence of each object upon all, and of all objects upon each, and to talk of a particular object's dependence upon certain other specific objects in a given situation, or in a series of them held together in an idea (as in the case of gold), are very different things.[3] To exhibit this matter of connections and disconnections between its objects constitutes in large measure the central business of chemistry.

Objects, as we thus perceive, are not without independence. Independence is further assured and made obvious when we recognize how, in this dynamic world of connections and disconnections, objects are seen to be as effective in reinforcing each other and in preserving each other intact as they are in building each other up or in destroying each other; and, in still other cases, not only disdain to enter into certain effective relations, but even block effective relations which might otherwise have come into existence. It is to chemistry with its combining laws of substances that we again may turn for the most precise formulation of each statement here made.

These empirical observations in support of an independence acquire additional confirmation when considering abstract relativity in the light of the principle of conservation. Here we learn that objects or terms may displace and modify each other, but that the extinction of one term has its equivalence established in the creation of others. That is, abstract relativity, in contradiction to the principle of conservation, would reduce a certain term to zero by a total abstraction of it from the rest of the universe; but it fails to note that this abstraction of one of its terms argues some change or other of the remaining terms; and, if we adhere to the principle of conservation in its abstract form (a thing I would myself avoid), such changes may be maintained as affecting all of the remaining terms. These facts then, taken together, even in this summary

[3] Once recognize this distinction in its full effect, and we will be found to have an empirical foundation for real change and novelty in the world; for the organization of our world, whether of science or of every-day life, whether of man's or of nature's production, involves such an *interchange* of effective elements, that many of them never yet have been and perhaps never will be brought together. Many of them, of course, have been; but herein lies the sphere and foundation of real novelty or creation in our world, especially if we do not in this connection forget our chemistry and Hume's special teaching that the incommensurable issues from every new combination of objects or elements.

statement of them, point to the conclusion that relativity (in its identity with the scientific postulate of general dependence) does *not* imply a total extinction of independence "into mere relations to other things, themselves equally characterless and elusive." Rather does the contrary hold true; without dependence there is no independence. How long, for example, would an object like "gold" preserve or realize its full independence (integrity) when the idea of gold, as generally recognized, and, in particular, as defined by a chemist, involves a host of qualities and properties which demand time and the most varied situations for their realization? Furthermore, without insinuating an arbitrary or biased standard of some kind or other, the qualities and properties it reveals in one situation are as truly to be considered of its nature as those it reveals in other situations. If a selection is made among them, the selection would intimate the existence of some new principle of control. But whatever the principle of organization that a specific object may involve, the fact remains, as in general true, that no single situation of actual existence can reveal or exhaust an object's total actuality, that is, all its possible phases or qualities.[4] It is in its very nature a mutiplicity viewed spatially or temporally. Hence when synthesized into a whole, our object assumes the character of a construct. But to preserve the independence (integrity) of such objects demands dependence. In so far, then, as we remain strictly empirical, and further, strictly adhere to our scientific postulate of general dependence, the following form of relativity seems the more permissible one: objects reveal themselves differently in different situations, and in different situations are capable of revealing qualities absolutely incompatible with each other.

But a mere proof of an *independence* is not a definition of *things;* for whatever other notion may be implied in our conception of a thing, a specific inclusion and exclusion of parts is inevitable to its specific and distinct recognition. Terms of some kind we must adopt, and in this conclusion we are justified by the relative independence of terms amid dependence and change; and it is to those aspects of reality where a term's substantiation may be gleaned, that realism is especially disposed to turn in defense of its demand for solid things. But *what* terms we must adopt, depends upon the nature of the problem in hand, and it is in this "what" aspect of terms that realism so commonly proves disappointing.

Turning attention, now, to this phase of the problem, the point

[4] The appreciation of this fact negates the idea of an Absolute of a completed reality. Time and varied situations are demanded in the realization of any complex reality, and the more complex the reality, the more largely does this demand apply.

of contact between relativity and realism opens up anew. Do objects exist in nature with a prefixed measure and boundary? If they do, what is the prefixed measure and boundary of every object? If they do not, what controls us in affixing such measure and boundary to objects as they acquire, however variable or identical the measure and boundaries are that are assigned to the objects about us? Thus we might ask: why does the average man and the physical chemist regard water, vapor, and ice as *three* things instead of following the lead of the general chemist in regarding them all as mere aspects of *one* and the same thing under varied conditions; namely, as aspects of H_2O, or all alike, including hydrogen and oxygen, as mere functions of weight? Relativity drives this problem home; realism, however, can not with impunity ignore it; for a mere reiteration that objects exist in nature with prefixed measure and boundary is no rational proof that such is the case, however much ultimately an agreement may manifest itself between the rational proof and the inarticulate protestations of common sense or good sense. I affirm, therefore, that realism and relativity in their common search for *things* (as previously in their common search for independence) focus in a common problem, the problem of the one-and-the-many.

To postulate an inherent essence (substance) in explanation of this problem will not help us, for in that case we postulate we know not what, as tradition from Locke through Kant has so fully taught. And where an object changes, and all objects do and can be made to change to an indefinite degree, the question arises to which group of its changing qualities shall I hold as representing *the* group of a particular object. Now chemistry has its own solution of the problem in its elementary substances just as modern physics has its present solution of it in electrons. As for philosophy (to quote from a recent book), "I do not say that it is impossible to solve the problem of the one-and-the-many, . . . but up to the present time no solution has been given."[5] Modern realism with its elusive and protean conception of an object would surely not offer its solution as a solution of the problem in hand. I wonder how long a chemist or a physicist would pause to receive the instruction thus offered.

Suppose, then, we accept the verdict of chemistry as provisionally valid. Our terms in that case are the elementary substances. The chemist regards them as irreducible and he regards them as constructs,—as specific groups of itemized properties demanding time and the most varied situations for their complete realization. But to speak of them as (1) irreducible and as (2) constructs involves a possible confusion. In this dual claim the one-and-the-many prob-

[5] Russell, "First Course in Philosophy," 1913, page 90.

lem is as obviously evoked as in any other quarter. The difficulty is further accentuated when we read chemistry in the light of the electron as ultimate, whereby, as affirmed by many of the leading physicists of to-day, the electron and not the elementary substance becomes the desired term. And thus the issue of the one-and-the-many problem continues and, for all we know, always will continue. But the ideal simplicity of the "term" (and this is important), the special aim of both of these sciences, is not of necessity the only ideal. Often we aim to know the effect of a complex term upon other complex terms. For example, when I make lemonade I do not combine elements that are simple from the standpoint of chemistry, but I combine the complex thing "water" with the complex things "sugar" and "lemon juice." Or when I seek the effect of a violin concerto upon an audience, I do not resolve the complex things—violin box, strings, bow, the air, the player, etc.—into parts that are elementary from the standpoint of chemistry, physics, psychology, or what not, but I deal directly with the interdependence or relations of parts exceedingly complex. Hence all we seem entitled to affirm thus far is, that the scientific postulate, in so far as it signifies so-called *conditions* and so-called *results,* presupposes the existence of terms at every step of its operation. Terms thus come and go; but with terms in some form or other we begin and with terms in some form or other we end. Hence the realist in his assertions that complex things are or may be ultimate, has, from the standpoint of relativity, here hit upon a bigger truth than he in his reasoning may have actually known; for its background is a clearly accepted pluralism and a clearly accepted relativity. The truth, however, of the ultimate character of complex things (under conditions) is impregnable. In addition the thing a relativist in the rôle of a pluralist would emphasize is, that no set of conditions with its results has in its conditions or in its results a *monoply* upon reality, whether in one case we adopt the electrons of physics as the foundation or limits of the objects involved, or the elementary substances of chemistry, or a sensory *quale.* Under conditions, then, as Locke himself wrote, "an army, a swarm, a city, a fleet are things as perfectly one as a ship or an atom."

Consistently hold to the fact, then, (1) that unity, under conditions, may be validly assigned to any object, from an electron to the universe in its totality, and (2) to the principle that given conditions produce a given result—then, whatever the conditions, the result *is* reality. Break up that result into parts, if you choose, or synthesize the given result with other results and, together, organize them into a larger result; in either case we have *reality,* of which common practise offers certification. For if it be once admitted that

objects exist in nature with no prefixed, natural, or inevitable unity or boundary of their own, then one boundary of them is no more true in the abstract than another, whether we proceed by the way of analysis to a pale and vapid *quale,* or by way of a synthesis advance to the Absolute of our objective idealists. Hence neither a *Ding an sich* nor a phenomenalism really has any meaning from a relativistic standpoint. A thing *is* what it reveals itself to be in any given situation, or, by a process of construction, *is* what it was found to be in a series of situations; and it logically remains entirely beside the issue whether a human organism, with its source for varied determination, constitutes a part of a situation or whether other objects do. For no set of conditions with its specific result, in the abstract, I repeat, has a prerogative or monopoly upon so-called reality. Constructs enter our world at every point, whether mechanical, industrial, artistic, scientific, or what not, and any one of them in its simplicity or in its complexity may become our real and actual point of departure in thinking as in action.

The length at which some writers, as Bergson for example, would decompose our ultimate units of knowledge because they are capable of a varied formulation or, in other cases, because they are capable of an indefinite analysis and decomposition, is not without error. Limits in these matters exist that Bergson and his confrères in anti-intellectualism are loath to admit or recognize. Thus we may add in the way of a refutation of these abstract claims that the *in*decomposable character of elements in chemistry is a far more rigid trait than their decomposable character, although some of the elements once thought of as indecomposable may slowly give way to a resolution coincident with the slow refinement of a complex technique elaborated in connection with them. Or again, in the case of our sense organism, is rigidity in and susceptibility toward the number, kind, and range of our sensations the conspicuous thing, or is it the opposite that is true? The answer is evident. Moreover, as stated, terms of an ideal simplicity—elementary substances, the electron, the *quale* of an immediate experience,—are not the only limits in the organization of our experiences. In the case of our violin concerto, limits present themselves that on their face are conspicuously complex. In fact, what would become of most of our sciences, in particular ethics, sociology, mechanics, if this were not the case? Limits exist, therefore, whether we move up or down for our unit; limits which, under conditions, make now one unit the basis of a synthesis and then some other.

When a thinker, therefore, in harmony with the scientific postulate of general dependence knocks at such limits, but refuses in his actual thinking to be controlled by them, he parts company with

what is empirical, to launch forth at will in a sphere of delightful (!) abstraction. Secondly, if he recognizes but *one* such limit, as Professor Lovejoy, for example, gives expression to in his sensory *quale* as the experienced datum, he is an absolutist who sees but one possible aspect of reality and mistakes it, however partial or fruitless, as adequate in its resolution of the whole universe with all its inherent variety and complexities. But whether the thing is determined by one *quale* or by a multitude of them, or whether the *quale* is determined by the thing, he does not say. In either case there is presupposed a determination of the *quale* or of the thing that is openly at issue. Thirdly, if as the realist does, a thinker (1) mistakes a mere assumption of an independence for its proof, or (2) mistakes an undefined independence for a defined *thing,* or (3) conceives of an independence apart from dependence, we might in each case be thought to have a doctrine of reality, but they would be doctrines of such value as I leave with my readers to judge.

I have sought to prove that relativity, when identified with the scientific principle of general dependence in its conjunction with so-called uniformity of nature, recognizes a variety of limits, which, as long as they remain practical, are actual; and that such limits are as apt to present themselves validly in the form of an obviously complex synthesis as in the direction of a minute analysis. Physics, chemistry, and the theory of the sensory *quale,* in contrast with realism, to my mind, more clearly define the nature of the limits they involve, but they differ from realism in a less general declaration for independence and pluralism. Finally, that a relativity that is empirical and pretends to adequacy must incorporate such independence and pluralism in its formulation just as obviously as a realism must recognize the postulate of general dependence (relativity) if it would be scientific. And further, that realists no less than relativists, before they can talk of *things* in any proper sense of the term, must both alike turn to the one-and-the-many problem as a common heritage of a metaphysical difficulty, and that it is in connection with this problem that I see hope for a fuller and clearer recognition and definition of the varied limits involved in the varied organization of our very complex and evolved experience.

Not in the conjunction of realism and relativity, therefore, but in their supposed incompatibility, do I see philosophical confusion.

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REVIEWS AND ABSTRACTS OF LITERATURE

The Problem of Christianity: Lectures delivered at the Lowell Institute in Boston, and at Manchester College, Oxford. JOSIAH ROYCE. New York: The Macmillan Company. 1913. 2 Vols. Pp. xlvi + 425; 442.

Some years ago the present writer chanced to be in Cambridge at the time when Professor Royce was delivering his lectures on the "Essence of Christianity" which were subsequently published in the *Harvard Theological Review* under the title, "What is Vital in Christianity." The lectures were delivered upon the invitation of the Students' Christian Association which, if my memory is not at fault, suggested the topic. Chancing to meet Professor Royce at the time, I expressed my satisfaction that we were to have the benefit of his thought upon so practical a theme. Royce shook his head. "I did it," he said, "because I promised, but I will never take such a subject again; it is too concrete for my taste."

Readers of the two substantial volumes in which Professor Royce has embodied his views upon the problem of Christianity must rejoice that his excursion into the realm of the concrete has not proved so isolated an experience as he anticipated. We are all the richer for sharing his thought on a theme which has proved of perennial interest to humanity, and we feel safe in assuming that he himself has found the hours spent in studying the origins of Christianity, and thinking over again the questions which its existence raises, an enlarging and rewarding experience. Those who, like the present writer, have made it their life-work to study the problems of Christian theology welcome the entrance into this well-cultivated field of an independent thinker who brings the virility of mind, the broad outlook, and at the same time the earnest religious spirit which characterize the distinguished philosopher who so ably carries on the tradition of idealism in the oldest of our American universities.

Yet it must be confessed that the spirit which uttered itself in the remark which was quoted in our opening paragraph is still in evidence in the volumes under review. Like Hegel, whom Professor Royce follows in his attempt to give a philosophic interpretation of the essence of Christianity, the American idealist believes that absolute truth is revealed in history, but he is equally sure that you must not identify it with any particular historical phenomenon, even if that phenomenon be one so august and sacred as the Christian religion itself. To the man who has the insight to see it, truth is present everywhere and always as the inner meaning of that which, looked at from the outside, appears simply as a series of brute facts. What is needed above all things is the formula, and this it is the business of the philosopher to furnish us. How it is to be applied, what bearing it will have upon the special problems of the individual life, he leaves to others as no part of his professional concern.

Yet it would be a great mistake to confuse Royce's conception of the essence of Christianity with Hegel's. In spite of superficial similarities there is a real difference. Between Hegel and Royce there lies a mass of critical research devoted to the investigation of the facts of the Christian

religion. Professor Royce disclaims any right to speak at first hand of the questions under controversy in this criticism; yet he writes as an intelligent layman who has studied what the critics have to say and comes to his own problem with an independent understanding of their results. You feel, in reading his pages, that the religion with which he deals is really the Christian religion, not some independent construction of his own brain. Some Christians may feel—many will doubtless feel—that there are aspects of the Christian religion—and these fundamental aspects—which find no place in Royce's book, but what he gives they will recognize as a part at least of the religion they know, and they will feel in him, therefore, a kindred spirit. To take but a single illustration,—it would be difficult to find a more illuminating piece of analysis than the comparison which the author draws between Buddhism and Christianity in the seventh chapter of his first volume, a comparison equally just in its estimate of similarity as in its recognition of difference.

The book falls into two parts, independent yet complementary. The first, entitled the "Christian Doctrine of Life," is a study of the essence of Christianity considered as an objective historical phenomenon. The second, which the author entitles "The Real World and the Christian Ideas," asks the question how far the Christian solution of the problem of life can maintain itself at the bar of metaphysical inquiry. Thus the two parts of Royce's book cover substantially the ground which is discussed by contemporary German apologetics under the titles "Das Wesen" and "Die Wahrheit des Christentums."

The plan has advantages and disadvantages. Its advantage consists in its clear recognition of the fact that the problem of Christianity is one of definition as well as of proof. Before we can tell what Christianity signifies we must know what it is, but this is in dispute not only among non-Christians, but among Christians. It is methodically correct, therefore, to isolate this question for independent discussion as Royce does.

On the other hand, the method has disadvantages, in that it separates matters that belong together. The definition of Christianity involves among other things a description of the beliefs of Christians about ultimate realities, notably about the nature of God. But this question, fundamental for the understanding of every religion, Royce passes over with only incidental reference,[1] reserving his discussion, so far as he gives one, for the second part of his treatise. The result is a description of Christianity which omits altogether what most Christians would regard as the heart of their religion. Royce himself admits this and defends his method on the ground that Christians have believed many things in the past which we all recognize to-day have been outgrown. But whether this can apply to a conception so fundamental as that of God is a question which requires fuller discussion than our author has given it.

To begin with the first problem, that of the essence of Christianity, it may be premised at the outset that Professor Royce believes that the quest of such an essence is legitimate and important. If the world has a meaning, if history is more than a mere succession of events without inner

[1] *E. g.*, Vol. I., pages 202, 205.

relation and spiritual significance, then a great historical phenomenon like Christianity must have some significance which can be discovered by the sympathetic and reverent student. History, as Royce conceives it, is more than a science. It is a philosophy, a search for the inner unity which connects events and makes them luminous and meaningful to the instructed observer.

There are, however, two possible methods of interpreting an historical phenomenon like Christianity, each of which has its advocates. You may find your essence revealed at some particular point of time, identifying it either with the teaching of the founder, the content of the Bible, or the dogma of the church, as the case may be. Or you may find it in some spirit which runs through the whole course of history and whose full secret is only gradually apprehended. You may say of Christianity, It is the religion of Christ, meaning the religion that Christ taught, or you may say, It is the religion of Christians, meaning that it is the experience into which Christians have progressively entered as they have tried to understand and interpret Christ. Royce's view is of the latter kind. Like Schleiermacher and Ritschl he takes his stand within the consciousness of the Christian community, which he regards as the real creator of Christianity. He does not think it possible to press back of this consciousness and to recover some more primitive form of faith which, when recovered, must henceforth be our test of true religion, nor indeed is he greatly interested in the attempt. The church rather than Christ is the constitutive idea of Christianity, and the church is only another name for the ideal community, which is the unifying principle of philosophy and religion alike.

Holding this view, it is not strange to find Professor Royce indifferent to the results of recent critical study of the life of Jesus. Of the founder of Christianity he speaks with great respect and of many of his reported teachings he gives a sympathetic and in many respects an illuminating interpretation (*e. g.*, Jesus's teaching concerning love), but as to his own place in the religion that bears his name he professes himself agnostic. It is not simply that he feels the difficulties raised by modern criticism,—that we could well understand,—but that the picture of Jesus as presented in the Gospels, the picture of a particular individual embodying a definite ideal of service and brotherhood and love, finds no natural place in his philosophy. In contrast to Hegel, who insists that the ideal personality who incarnates the principles of his philosophy must at some definite time appear, Royce does not expect "any human and visible triumph of the ideal in history" (Vol II., page 430). All that we can hope for is an approximation toward that ideal, the assurance that somewhere and somehow (not here and now) it is realized in the eternal.

Passing from method to result, we find that our author discovers the essence of Christianity in three controlling ideas: the idea of the church, or the universal community through loyalty to which alone the individual realizes his destiny; the idea of sin, or the moral contradiction in which the individual finds himself necessarily involved because of the conflict between his own will and that of the community, and the idea of atone-

ment, or the saving deed of the community or its chosen representative through which this disharmony is overcome and the individual restored to his true place in the communal life. The ideas which underlie the discussion have already been anticipated in earlier works, notably in "The Philosophy of Loyalty," and "The Sources of Religious Insight." But the treatment forms a unity and can be followed without reference to any of the author's previous works.

Fundamental for Royce's thought is the conception of the community as an independent entity having a unity of its own as definite and as empirically verifiable as that of an individual personality. This conception to which Royce had come on independent grounds in his "Philosophy of Loyalty" he holds to be the central reality for the Christian consciousness. The Pauline church, or, as he paraphrases it, the beloved community, is in fact the community of the loyal. This is the body of Christ, the sphere in which his spirit expresses itself, the end in devotion to which every individual finds his true self fulfilled.

This view of the church explains Royce's view of the second of the great Christian conceptions, that of sin. The supreme sin, according to our author, is disloyalty, treachery to the cause of the community. But this he finds to be the inevitable result of the conflict between the individual's wants and desires and the claim of the community of which he is a part. The more highly developed the individual, the more conscious he is of capacities and desires that are inherently worthy, the more he rebels against the restraints which the community puts upon him, the more his experience approximates that of Paul in the seventh chapter of Romans. What is the remedy for this situation? It is the remedy which Paul discovered, namely, the existence of a new and higher social order, in loyalty to which the individual can find his highest desires fulfilled. Such a community is the church of Christ. Through loyalty to it alone salvation is possible. But what of those who have proved false to this new and higher allegiance? Such would seem to have committed the unpardonable sin, a sin for which even if others could forgive, they could never forgive themselves. What has religion to offer to those who have been disloyal to the cause of loyalty? This brings us to the third of the ruling ideas of Christianity which Royce discovers, namely, that of atonement.

Professor Royce finds in the Christian emphasis upon the atonement one of its chief claims to our acceptance. "The human aspect of the Christian idea of atonement," he tells us, "is based upon such motives that if there were no Christianity and no Christians in the world the idea of atonement would have to be invented before the higher levels of our moral existence could be fairly understood (Vol. I., page 271). It is quite true that the particular theories of atonement which meet us in the official theology of Christianity are unsatisfactory. But this is because the problem has been conceived in too superficial and external a way. Either, like the penal theory, the deed has been isolated from its consequences and atonement found in some form of legal substitution; or like the moral theories, a moral transformation of character in the individual has been considered sufficient. But what is needed is something more profound

than either, such a dealing with the tragedy of sin that the world shall be the better for the fact that it has happened. What if it should appear that the traitor's deed of disloyalty prove the occasion for some corresponding deed of love so wise, so gracious, so winning in the beauty of its unselfish adaptation to the tragic situation that not only others, but even the traitor himself who is responsible for the evil that called it forth must recognize that the world is the better for its having been done. Such a deed, could it be done, would fulfil the conditions of a real atonement, and such, in the opinion of our author, is the remedy which Christianity proposes for the tragedy of sin.

In spite of the abstract form in which it is cast, one must recognize in the analysis a true insight into the genius of historical Christianity. It is the more to be regretted that the author should have passed so lightly over other ideas to which most Christians would attribute equal if not superior importance, namely, that of God, the ultimate object of Christian faith, and Jesus Christ, the historic mediator of his revelation.

Of the reason for the first omission I have already spoken. The explanation of the second appears in the second and metaphysical part of the book. It is here made clear that the reason why Royce gives no independent doctrine of God is that he identifies God with the community, which from a different point of view is only another name for the living Christ or the Holy Spirit of historic Christian theology. This identification is possible because the community which is the object of Christian loyalty is not the empirical society we call the church in any one of its many conflicting forms, but a noumenal reality which as completely transcends present (and for that matter all future possible) experience as the God of Kant himself.

Royce's justification for his identification of the ultimate reality, or God, with the ideal society, has two parts. The first develops the idea of the community as the ultimate metaphysical reality; the second gives reasons for the belief that such a reality actually exists. The former occupies chapters nine and ten, which deal with the Community and the Time Process and the Body and its Members; the latter is discussed in chapters eleven to fourteen, which treat of the significance of interpretation for our knowledge of reality. Two concluding chapters draw certain practical conclusions.

Of the first point I can speak briefly. Royce makes use of Wundt's "Studies in Social Psychology" to defend the thesis that the community is a true unit in that it has a common past and a common future. It has a common past because the same historical event may enter into the consciousness of each of its members who, through this common inheritance, become what Royce calls a community of memory. It has a common future because each of its members may work for the same ideal, and this common purpose constitutes them a community of expectation.

We have no criticism to make of this interesting discussion other than to remark that it opens the way for an interpretation of history far more concrete and vital than that which Royce has given. If the possession of a common past is essential to the existence of a community in the sense in

which Royce defines it, then it would seem that the specific historic facts which constitute that past must bulk more largely than they do for our author, and the Christian's backward look to Jesus as the founder of the Christian community can not be dismissed in the summary manner in which he dismisses it.

More interesting, because more original, is Royce's use of the theory of interpretation as a justification of his metaphysical position. For this theory he confesses his indebtedness to Mr. Charles Pierce, that fruitful source of suggestion to American philosophers. But the use which Royce makes of it is his own. According to this view there are not two methods of knowledge as ordinarily supposed, perception and conception, but three, perception, conception, and interpretation. Perception is concerned with that which is immediately given in experience; conception with general ideas reached by a process of abstraction; but interpretation is the process by which we share what we have experienced with other persons for purposes of common action. Both perception and conception admit of being stated in individual terms, but interpretation is essentially social. It takes for granted the existence of a society of persons and would be meaningless without it.

Here again there is much that is interesting in Professor Royce's discussion, over which it would be a pleasure to linger. But we are concerned here primarily with its bearing upon his argument as a whole. If we understand him aright, what he gives us is a new variety of the ontological argument. We must assume, so the argument runs, that the reality which is implied in the very processes of our knowledge has objective existence. As used in the older forms of the ontological argument this reasoning, even if its validity were admitted, could give us at most a highly abstract result such as Kant's "Ens Realissimum." But as used by Royce in the light of his doctrine of interpretation it gives us the most concrete of all results, namely, the beloved community itself. If all our knowledge involves the process of interpretation, and interpretation is meaningless without the existence of the community with whose members insight is shared, then we must believe that the nature of ultimate reality is social, which is the thesis which is to be proved.

A corollary of the doctrine of interpretation is the doctrine of signs. A sign is any object which admits of interpretation, *e. g.*, "a word, a clock-face, a weather vane, or a gesture" (p. 283). Not only art and literature, but science and common sense, are constantly making use of signs. Experience itself may be defined as "a realm of signs" (p. 289), and history is constantly presenting us with objects which through interpretation give us insight into the nature of reality. What we all recognize as valid in particular instances Royce would extend to the world as a whole. "The world is the community. The world contains its own interpreter. Its processes are infinite in their temporal varieties; but their interpreter, the spirit of this universal community,—never absorbing varieties or permitting them to blend,—compares and through a real life interprets them all" (p. 324).

One must regret that Professor Royce has not followed this clue

further and given us a classification of signs with special reference to the signs of which religion makes use, and especially the particular religion which he is studying. This would enable us to test the correctness of his previous analysis and discover why he omits from his catalogue so many of the ideas which have played so large a rôle in historic Christianity. But this would take him back into the realm of the concrete, from which he has so happily escaped. As it is, we must be content with the general principle without application in detail.

In one of his closing chapters Professor Royce, by a use of the creative imagination, not uncommon among philosophers, calls back from the grave a Pauline Christian and, transporting him across the lapse of years into the new world of modern science, presents him with a copy of "The Problem of Christianity," in order to discover how far he will recognize in it the essence of his own religion. As could hardly be otherwise under the circumstances, the test is successful and the position of the author triumphantly vindicated. For a moment, to be sure, our convert hesitates, but being at heart a gentlemen as well as a philosopher, he recognizes in Professor Royce a kindred spirit and soon finds himself at home in his new environment.

If the reviewer might venture upon a like liberty he would suggest a repetition of the experiment. There were more types of Christians in the Pauline community than one, and it may be that a second visitor would look at the matter from a somewhat different angle. He might address this new interpreter of Christianity as follows: "This religion that you call Christianity has much that attracts me. I recognize my own experience in your description of sin and of salvation. I respond with enthusiasm to your plea for loyalty, but there is one thing I miss, and that is a personality fitted to call forth my loyalty. Such a personality I knew in Jesus Christ, the founder of the church to which I gave my allegiance. But in this sublimated community, which I am not at liberty to identify with any existing society, I look in vain for my Master and my Lord. What I need is not the divine in general, but the divine as revealed in the human, and such a revelation I find in the person of Jesus. I rejoice in your protest against reducing Christianity to a mere ethical system. I welcome your re-emphasis upon the universal and metaphysical elements in my faith, but why need the emphasis of the one involve the sacrifice of the other? Christianity, to be sure, is religion, not ethics; but it is ethical religion, and the reason why Christ must ever hold the central place in Christian faith is the fact that he embodies to the imagination of men in the picture of a human life principles universally applicable. He is the sign by which we interpret the universe and discover the nature of the social order after which we are to strive. Put back Christ into Christianity and I shall find myself at home in the beloved community whose ideals you have so fascinatingly sketched."

Whether our resuscitated Pauline Christian would speak thus I do not know, but I am sure this is a sentiment which will find an echo in the heart of many a modern Christian as he lays down Professor Royce's book.

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JOURNALS AND NEW BOOKS.

REVUE PHILOSOPHIQUE. May, 1914. *Sous Quelles Conditions le Mysticisme Est Légitime* (pp. 449–481): JULES DE GAULTIER. – Only those forms of mysticism are legitimate which "do not enter into conflict with phenomenal experience, but, on the contrary, confirm, explain, and transfigure it, for it is mysticism's accord with experience which in the end determines its legitimacy in the domain of the *possible*." *Vers l'Intuition Expérimentale de l'Electron* (pp. 482–505): A. REY. – This, the last article of the series, reviews the evidence obtained since 1908, and arrives at the final conclusion that "experimentation has put beyond doubt the existence of an elementary electric charge." *Contribution a l'Étude du Sentiment Amoureux* (pp. 506–526): N. KOSTYLEFF. – A study of extra-genital erotism, and of its associations with mental images. *Revue Critique:* FR. PAULHAN. *Analyses et Comptes Rendus.* H. Reverdin, *La Notion d'Expérience d'après William James:* FR. P. Dr. Werner, *Le Pragmatisme de James et de Schiller:* FOURET. A. de Gramont-Lesparre, *Les Inconnus de la Biologie Déterministe;* Dr Agostino Gemelli, *Il Metodo degli Equivalenti:* G. L. DUPRAT. James Mark Baldwin, *History of Psychology:* M. SOLOVINE. G. F. Arnold, *Psychology Applied to Legal Evidence:* G. L. DUPRAT. Carlo Spanzini, *Die Fortschritte der Voelkerpsychologie von Lazarus bis Wundt:* DR. JANKÉLÉVITCH. Paul Souriau, *L'Esthétique de la Lumière:* LUCIEN ARRÉAT. Andre Fauconnet, *L'Esthétique de Schopenhauer:* A. JOUSSAIN. E. Guyot, *Essai sur la Formation Philosophique du Poête Arthur Hugh Clough:* FOURET. *Revue des Périodiques.*

ARCHIVES INTERNATIONALES DE NEUROLOGIE. July, 1914. *Traitement de l'État de Mal Épileptique* (pp. 1–3): HENRI DAMAYE. – In the condition known as "*status epilepticus*," the administration of bromides is a very unsatisfactory method of treatment; and a solution of chloral hydrate in hot milk, introduced into the stomach by means of the œsophageal tube, should be substituted. *Sténose Congénitale de l'Aorte et des Artères avec Epilepsie* (pp. 3–14): A. OBREGIA, C. J. URECHIA & A. POPEA. – Congenital aortic stenosis may be regarded as one of the causes of epilepsy. *Ptoses Viscérales et Hallucinations Visuelles* (pp. 14–22): HENRI AIMÉ. – Visual hallucinations are often met with in cases of visceral ptosis, and are due to the traction on the abdominal sympathetic ganglia by the different viscera. *Revue des Sociétés. Analyses Bibliographiques.*

Allen, Edward. Wayside. Boston: Sherman, French, and Company. 1914. Pp. 159.

Calkins, Mary Whiton. A First Book in Psychology. Fourth Revised Edition. New York: The Macmillan Company. 1914. Pp. xxi + 428.

Edelman, Philip E. Experiments. Minneapolis, Minn.: Philip E. Edleman. 1914. Pp. 256. $1.50.

Garritt, E. F. The Theory of Beauty. New York: The Macmillan Company. 1914. Pp. v + 304. $2.00.

Keyser, Cassius J. Science and Religion: The Rational and the Super-Rational. New Haven, Conn.: Yale University Press. 1914. Pp. 75. 75 cents.

NOTES AND NEWS

THE *Rivista di Filosofia Neo-Scolastica* attaches the following notice to its number for September 20: (1) If the current of events permit, the next number will be wholly dedicated to Roger Bacon. Distinguished Italian, German, and English writers will collaborate to it. (2) We invite all those who are interested in philosophy to send us their visiting cards. These we shall forward to the President (Rector) of the Leo XIII Institute of Philosophy of the University of Louvain, to express to him our sympathy for the colleagues of that glorious institution in their sad bereavement. (3) As the storm rages over Europe in this fateful hour, we, both as scholars and as believers, can not refrain from urging our friends to unite with us in prayer. And this for two reasons: To invoke God, in whose hands are the destinies of man that he may in his mercy turn the course of events, and out of this great evil bring the blessing of peace upon our hearts and restore comfort to our souls. And, secondly, to invoke the protection of God in behalf of our numerous friends who are at present engaged with the belligerent armies. Many students of scholastic philosophy are now fighting as soldiers of Belgium, Germany, and France, and from very few of them have we any news. May they return unharmed, after they have gloriously served their country, to the defense of our common heritage of learning.

DR. A. E. DAVIES, professor of philosophy in the Ohio State University, is away on a year's leave of absence. On account of the war he will be unable to visit the European universities, and has accepted the opportunities extended to him by the English universities for carrying out his original programme of work. His address is 44 Beechhill Road, Eltham, Kent, England.

THE New York University Philosophical Society met in the Students' Room of the university, Washington Square, on Tuesday evening, October 20. Professor Edward Gleason Spaulding, of Princeton University, addressed the meeting on the topic "A Defense of Reason."

MANY students of philosophy and sociology will be interested in the address of the president to the Anthropological Section of the British Association for the Advancement of Science, entitled, "A Study of Primitive Characters," delivered in Australia, and printed in *Science* for October 9.

FOUR lectures on "Heredity" were recently given by Dr. Sandwith at Gresham College. The first dealt with the subject from the purely scientific point of view and the remaining three discussed it in its social aspect, largely with reference to eugenics.